ADJOINT COHOMOLOGY OF GRADED LIE ALGEBRAS OF MAXIMAL CLASS

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Abstract. We compute explicitly the adjoint cohomology of two $\mathbb{N}$-graded Lie algebras of maximal class (infinite dimensional filiform Lie algebras) $\mathfrak{m}_0$ and $\mathfrak{m}_2$. It is known that up to an isomorphism there are only three $\mathbb{N}$-graded Lie algebras of the maximal class. The third algebra from this list is the "positive" part L_1 of the Witt (or Virasoro) algebra and its adjoint cohomology was computed earlier by Feigin and Fukhs. We show that the total space $H^*(\mathfrak{m}_j, \mathfrak{m}_j)$ is "almost" isomorphic to the completed tensor product $\mathfrak{m}_j \otimes H^*(\mathfrak{m}_j)$, $j = 0, 2$.

Introduction

A. Shalev and E. Zelmanov defined in [10] the *coclass* (which might be infinity) of a finitely generated and residually nilpotent Lie algebra $\mathfrak{g}$ as $cc(\mathfrak{g}) = \sum_{i\geq 1}(\dim(C^i\mathfrak{g}/C^{i+1}\mathfrak{g}) - 1)$, where $C^i\mathfrak{g}$ denotes the i-th ideal of the central descending series of $\mathfrak{g}$. Algebras of coclass 1 are also called algebras of *maximal class*. In finite dimensional case the notion of Lie algebra of maximal class is equivalent to the notion of *filiform* Lie algebra introduced by Vergne in [11]. In the study of filiform Lie algebras the $\mathbb{N}$-graded filiform Lie algebra $\mathfrak{m}_0(n)$ plays a special role. It is defined by its basis $e_1, \dots, e_n$ and its non-trivial commutator relations: $[e_1, e_i] = e_{i+1}, i = 2, \dots, n-1$.

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The natural infinite dimensional analog of $\mathfrak{m}_0(n)$ is a $\mathbb{N}$-graded Lie algebra $\mathfrak{m}_0$ of maximal class. It follows from [2] (see also [10]) that up to an isomorphism there are only three $\mathbb{N}$-graded Lie algebras of maximal class (or infinite dimensional filiform Lie algebras):

$$\mathfrak{m}_0, \ \mathfrak{m}_2, \ L_1,$$

where L_1 denotes the "positive" part of the Witt or Virasoro algebra and $\mathfrak{m}_2$ is defined by its infinte basis $e_1, e_2, \dots, e_n, \dots$ and the structure relations:

$$[e_1, e_i] = e_{i+1}, i = 2, \dots, \quad [e_2, e_j] = e_{j+2}, j = 3, \dots.$$

We recall that the scalar cohomology $H^*(L_1)$ was calculated in [7]. The cohomology $H^*(\mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2)$ were calculated in [3].

The next natural question is to calculate the cohomology of these algebras with coefficients in the adjoint representation, which is very important due to applications in the deformation theory. The adjoint cohomology $H^*(L_1, L_1)$ was calculated in [1]. In the present article we explicitly compute the cohomology $H^*(\mathfrak{m}_0, \mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2, \mathfrak{m}_2)$. One has to remark that the first cohomology spaces $H^1(\mathfrak{m}_0(n), \mathfrak{m}_0(n))$ and $H^1(\mathfrak{m}_2(n), \mathfrak{m}_2(n))$ were computed in [11, 9] and this result implies the answer in the infinite dimensional case. The structure of $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ and $H^1(\mathfrak{m}_2, \mathfrak{m}_2)$ was rediscovered recently in [4, 5]. Vergne's algorithm for constructing of a basis of $H^2(\mathfrak{m}_0(n), \mathfrak{m}_0(n))$ can be easily generalized to the infinite dimensional case.

The paper is organized as follows. In Sections 1–2 we review all necessary definitions and facts concerning Lie algebra cohomology and Lie algebras of maximal class. We recall a natural filtration of the adjoint complex $C^*(\mathfrak{g}, \mathfrak{g})$ and the corresponding spectral sequence E_r for an

arbitrary $\mathbb{N}$-graded Lie algebra $\mathfrak{g}$. This spectral sequence was successfully used by Feigin and Fukhs [1] for the computation of $H^*(L_1, L_1)$. This spectral sequence is the main technical tool that we use for the computation of $H^*(\mathfrak{m}_0, \mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2, \mathfrak{m}_2)$. The first term of this spectral sequence is isomorphic to the completed tensor product $\mathfrak{g} \otimes H^*(\mathfrak{g})$, where $H^*(\mathfrak{g})$ stands for the scalar cohomology of $\mathfrak{g}$. In some sense we show that the differentials of higher orders of our spectral sequence are "almost" trivial.

In the sections 3,5 we recall the structure of the cohomology $H^*(\mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2)$ with trivial coefficients obtained in [3].

In the sections 4,6 we apply the spectral sequence considered above to the computation of $H^*(\mathfrak{m}_0, \mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2, \mathfrak{m}_2)$ respectively. Namely we give the explicit answer in terms of formal series of the infinite basis of cocycles $\{\Psi_{I,r}\}$ in the case of $\mathfrak{m}_0$ and $\{\Phi_{J,q}\}$ for the algebra $\mathfrak{m}_2$.

1. Lie algebras of maximal class and filiform Lie algebras

The sequence of ideals of a Lie algebra $\mathfrak{g}$

$$C^1\mathfrak{g} = \mathfrak{g} \supset C^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset \dots \supset C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}] \supset \dots$$

is called the descending central sequence of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called nilpotent if there exists s such that:

$$C^{s+1}\mathfrak{g} = [\mathfrak{g}, C^s\mathfrak{g}] = 0, \quad C^s\mathfrak{g} \neq 0.$$

The natural number s is called the nil-index of the nilpotent Lie algebra $\mathfrak{g}$.

Definition 1.1. A nilpotent n-dimensional Lie algebra $\mathfrak{g}$ is called filiform Lie algebra if it has the nil-index $s = n-1$.

Example 1.2. The Lie algebra $\mathfrak{m}_0(n)$ is defined by its basis $e_1, e_2, \dots, e_n$ with the commutating relations:

$$[e_1, e_i] = e_{i+1}, \ \forall\, 2 \le i \le n{-}1.$$

Proposition 1.3. *Let $\mathfrak{g}$ be a filiform Lie algebra, one can remark that*

$$\sum_{i \ge 1} (\dim(C^i\mathfrak{g}/C^{i+1}\mathfrak{g}) - 1) = 1,$$

where in the sum only the first summand is non trivial.

Definition 1.4. A Lie algebra $\mathfrak{g}$ is called residually nilpotent if

$$\cap_{i=1}^{\infty} C^s\mathfrak{g} = 0.$$

Definition 1.5. A coclass of Lie algebra $\mathfrak{g}$ (which might be infinity) is a number $cc(\mathfrak{g})$ defined as $cc(\mathfrak{g}) = \sum_{i \ge 1}(\dim(C^i\mathfrak{g}/C^{i+1}\mathfrak{g}) - 1)$. Algebras of coclass 1 are also called algebras of maximal class or infinite dimensional filiform Lie algebras.

Example 1.6. Let us define the algebra L_k as the infinite-dimensional Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$ with a zero in $x = 0$ of order not less then $k+1$.

The algebra L_k can be defined by its infinite basis and commutating relations

$$e_i = x^{i+1}\frac{d}{dx}, \ i \in \mathbb{N}, \ i \ge k, \qquad [e_i, e_j] = (j-i)e_{i+j}, \ \forall\, i, j \in \mathbb{N}.$$

L_1 is a residual nilpotent Lie algebra of maximal class generated by e_1 and e_2.

We recall that $\mathbb{Z}$-graded Lie algebra W, defined by the basis $e_i,\ i \in \mathbb{Z}$, and relations

$$[e_i.e_j] = (j-i)e_{i+j} \quad \forall i, j \in \mathbb{Z},$$

is called the Witt algebra [6]. Hence, the algebra L_1 is the positive part $W_+ = \oplus_{i>0}(W)_i$ of the Witt algebra.

One can consider another examples of algebras of maximal class that are $\mathbb{N}$-graded Lie algebras.

Example 1.7. The Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ with commutator relations:

$$[e_1, e_i] = e_{i+1},\ \forall\ i \geq 2.$$

Example 1.8. The Lie algebra $\mathfrak{m}_2$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ and commutator relations:

$$[e_1, e_i] = e_{i+1}, \quad \forall\ i \geq 2; \qquad [e_2, e_j] = e_{j+2}, \quad \forall\ j \geq 3.$$

Theorem 1.9 ([2],[10])**.** *Let $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ be a $\mathbb{N}$-graded Lie algebra of maximal class. Then $\mathfrak{g}$ is isomorphic to one (and only one) Lie algebra from the three given ones:*

$$\mathfrak{m}_0,\ \mathfrak{m}_2,\ L_1.$$

2. Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ its linear representation (or in other words V is a $\mathfrak{g}$-module). We denote by $C^q(\mathfrak{g}, V)$ the space of q-linear skew-symmetric mappings of $\mathfrak{g}$ into V. Then one can consider an algebraic complex:

$$V \xrightarrow{d_0} C^1(\mathfrak{g}, V) \xrightarrow{d_1} C^2(\mathfrak{g}, V) \xrightarrow{d_2} \dots \xrightarrow{d_{q-1}} C^q(\mathfrak{g}, V) \xrightarrow{d_q} \dots$$

where the differential d_q is defined by:

$$\begin{aligned}(d_q f)(X_1, \dots, X_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \dots, \hat{X}_i, \dots, X_{q+1})) + \\ &+ \sum_{1 \le i < j \le q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}).\end{aligned} \tag{1}$$

The cohomology of the complex $(C^*(\mathfrak{g}, V), d)$ is called the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the representation $\rho : \mathfrak{g} \to V$.

In this article we will consider two main examples:

1) $V = \mathbb{K}$ and $\rho : \mathfrak{g} \to \mathbb{K}$ is trivial;

2) $V = \mathfrak{g}$ and $\rho = ad : \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$.

The cohomology of $(C^*(\mathfrak{g}, \mathbb{K}), d)$ (the first example) is called the cohomology with trivial coefficients of the Lie algebra $\mathfrak{g}$ and is denoted by $H^*(\mathfrak{g})$. Also we fixe the notation $H^*(\mathfrak{g}, \mathfrak{g})$ for the cohomology of $\mathfrak{g}$ with coefficients in the adjoint representation.

Let $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ be a $\mathbb{Z}$-graded Lie algebra and $V = \oplus_\beta V_\beta$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module, i.e., $\mathfrak{g}_\alpha V_\beta \subset V_{\alpha+\beta}$. Then the complex $(C^*(\mathfrak{g}, V), d)$ can be equipped with the $\mathbb{Z}$-grading $C^q(\mathfrak{g}, V) =$

$\bigoplus_\mu C^q_{(\mu)}(\mathfrak{g}, V)$, where a V-valued q-form c belongs to $C^q_{(\mu)}(\mathfrak{g}, V)$ iff for $X_1 \in \mathfrak{g}_{\alpha_1}, \dots, X_q \in \mathfrak{g}_{\alpha_q}$ we have

$$c(X_1, \dots, X_q) \in V_{\alpha_1+\alpha_2+\ldots+\alpha_q+\mu}.$$

This grading is compatible with the differential d and hence we have $\mathbb{Z}$-grading in cohomology:

$$H^q(\mathfrak{g}, V) = \bigoplus_{\mu \in \mathbb{Z}} H^q_{(\mu)}(\mathfrak{g}, V).$$

Remark. The trivial $\mathfrak{g}$-module $\mathbb{K}$ has only one non-trivial homogeneous component $\mathbb{K} = \mathbb{K}_0$.

Example 2.1. Let $\mathfrak{g}$ be an infinite dimensional Lie algebra with the infinite basis $e_1, e_2, \dots, e_n, \dots$ and commutating relations

$$[e_i, e_j] = c_{ij} e_{i+j}.$$

Let us consider the dual basis $e^1, e^2, \dots, e^n, \dots$. One can introduce a grading (that we will call the weight) of $\Lambda^*(\mathfrak{g}^*) = C^*(\mathfrak{g}, \mathbb{K})$:

$$\Lambda^*(\mathfrak{g}^*) = \bigoplus_{\lambda=1}^{\infty} \Lambda^*_{(\lambda)}(\mathfrak{g}^*),$$

where a subspace $\Lambda^q_{(\lambda)}(\mathfrak{g}^*)$ is spanned by q-forms $\{e^{i_1} \wedge \ldots \wedge e^{i_q},\ i_1 + \ldots + i_q = \lambda\}$. For instance a monomial $e^{i_1} \wedge \dots \wedge e^{i_q}$ has the degree q and the weight $\lambda = i_1 + \ldots + i_q$.

The complex $(C^*(\mathfrak{g}, \mathfrak{g}), d)$ is $\mathbb{Z}$-graded:

$$C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{\mu \in \mathbb{Z}} C^*_{(\mu)}(\mathfrak{g}, \mathfrak{g}),$$

where $C^q_{(\mu)}(\mathfrak{g}, \mathfrak{g})$ is spanned by monomials $\{e_l \otimes e^{i_1} \wedge \ldots \wedge e^{i_q},\ i_1 + \ldots + i_q + \mu = l\}$.

Let $\mathfrak{g} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ be a $\mathbb{N}$-graded Lie algebra. One can define a decreasing filtration $\mathcal{F}$ of the adjoint cochain complex $(C^*(\mathfrak{g},\mathfrak{g}), d)$ of $\mathfrak{g}$:

$$\mathcal{F}^0 C^*(\mathfrak{g},\mathfrak{g}) \supset \dots \supset \mathcal{F}^q C^*(\mathfrak{g},\mathfrak{g}) \supset \mathcal{F}^{q+1} C^*(\mathfrak{g},\mathfrak{g}) \supset \dots$$

where the subspace $\mathcal{F}^q C^{p+q}(\mathfrak{g},\mathfrak{g})$ is spanned by $p{+}q$-forms c in $C^{p+q}(\mathfrak{g},\mathfrak{g})$ such that

$$c(X_1,\dots,X_{p+q}) \in \bigoplus_{\alpha=q}^{+\infty} \mathfrak{g}_\alpha,\ \forall X_1,\dots,X_{p+q} \in \mathfrak{g}.$$

The filtration $\mathcal{F}$ is compatible with d.

Let us consider the corresponding spectral sequence $E_r^{p,q}$:

Proposition 2.2. *$E_1^{p,q} = \mathfrak{g}_q \otimes H^{p+q}(\mathfrak{g})$.*

We have the following natural isomorphisms:

$$\begin{gathered} C^{p+q}(\mathfrak{g},\mathfrak{g}) = \mathfrak{g} \otimes \Lambda^{p+q}(\mathfrak{g}^*) \\ E_0^{p,q} = \mathcal{F}^q C^{p+q}(\mathfrak{g},\mathfrak{g}) / \mathcal{F}^{q+1} C^{p+q}(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}_q \otimes \Lambda^{p+q}(\mathfrak{g}^*). \end{gathered} \tag{2}$$

Now the proof follows from the formula for the differential $d_0^{p,q} : E_0^{p,q} \to E_0^{p+1,q}$:

$$d_0(X \otimes f) = X \otimes df,$$

where $X \in \mathfrak{g}$, $f \in \Lambda^{p+q}(\mathfrak{g}^*)$ and df is the standart differential of the cochain complex of $\mathfrak{g}$ with trivial coefficients.

Remark. The spectral sequence considered above was used by Feigin and Fukhs [1] in their computations of $H^*(L_1, L_1)$.

3. Scalar cohomology of $\mathfrak{m}_0$

The cohomology algebra $H^*(\mathfrak{m}_0)$ was calculated in [3]. We will briefly recall some results from this article.

It were introduced two operators:

1) $D_1 = ad^*e_1 : \Lambda^*(e_2, e_3, \dots) \to \Lambda^*(e_2, e_3, \dots)$,

$$D_1(e^2) = 0,\ D_1(e^i) = e^{i-1},\ \forall i \geq 3, \tag{3}$$
$$D_1(\xi \wedge \eta) = D_1(\xi) \wedge \eta + \xi \wedge D_1(\eta),\ \ \forall \xi, \eta \in \Lambda^*(e_2, e_3, \dots).$$

2) and its right inverse $D_{-1} : \Lambda^*(e^2, e^3, \dots) \to \Lambda^*(e^2, e^3, \dots)$,

$$e^i = e^{i+1},\ D_{-1}(\xi \wedge e^i) = \sum_{l \geq 0} (-1)^l D_1^l(\xi) \wedge e^{i+1+l}, \tag{4}$$

where $i \geq 2$ and ξ is an arbitrary form in $\Lambda^*(e^2, \dots, e^{i-1})$. The sum in the definition (4) of D_{-1} is always finite because D_1^l decreases the second grading by l. For instance,

$$D_{-1}(e^i \wedge e^k) = \sum_{l=0}^{i-2} (-1)^l e^{i-l} \wedge e^{k+l+1}.$$

Proposition 3.1. *The operators D_1 and D_{-1} have the following properties:*

$$d\xi = e^1 \wedge D_1\xi,\ e^1 \wedge \xi = dD_{-1}\xi,\ D_1 D_{-1}\xi = \xi, \quad \xi \in \Lambda^*(e^2, e^3, \dots).$$

Theorem 3.2 ([3])**.** *The infinite dimensional bigraded cohomology $H^*(\mathfrak{m}_0) = \oplus_{k,q} H^q_k(\mathfrak{m}_0)$ is spanned by the cohomology classes of e^1, e^2 and of the following homogeneous cocycles:*

$$\omega_I = \omega(e^I) = \omega(e^{i_1} \wedge \dots \wedge e^{i_q} \wedge e^{i_q+1}) = \sum_{l \geq 0} (-1)^l D_1^l(e^{i_1} \wedge \dots \wedge e^{i_q}) \wedge e^{i_q+1+l}, \tag{5}$$

where $q \geq 1, \; I = (i_1, \dots, i_{q-1}, i_q, i_q+1), 2 \leq i_1 < i_2 < \dots < i_q$. *The multiplicative structure is defined by*

$$(6) \quad [e^1] \wedge \omega(\xi \wedge e^i \wedge e^{i+1}) = 0, \; [e^2] \wedge \omega(\xi \wedge e^i \wedge e^{i+1}) = \omega(e^2 \wedge \xi \wedge e^i \wedge e^{i+1}),$$

$$\omega(\xi \wedge e^i \wedge e^{i+1}) \wedge \omega(\eta \wedge e^j \wedge e^{j+1}) = \sum_{l=0}^{j-i-2} (-1)^l \omega(D_1^l(\xi \wedge e^i) \wedge e^{i+1+l} \wedge \eta \wedge e^j \wedge e^{j+1}) +$$

$$+ (-1)^{i-j+deg\eta} \sum_{s \geq 1} \omega((ad^* e_1)^{i-j-1+s}(\xi \wedge e^i) \wedge D_1^s(\eta \wedge e^j) \wedge e^{j+s} \wedge e^{j+s+1}),$$

where $i < j$, ξ *and* η *are arbitrary homogeneous forms in* $\Lambda^*(e^2, \dots, e^{i-1})$ *and* $\Lambda^*(e^2, \dots, e^{j-1})$, *respectively.*

Formula (5) determines a homogeneous closed $(q+1)$-form of the second grading $i_1 + \ldots + i_{q-1} + 2i_q + 1$. It has only one monomial in its decomposition of the form $\xi \wedge e^i \wedge e^{i+1}$ and it is $e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_q+1}$.

The whole number of linearly indepenedent q-cocycles of the second grading $k + \frac{q(q+1)}{2}$ is equal to

$$\dim H^q_{k+\frac{q(q+1)}{2}}(\mathfrak{m}_0) = P_q(k) - P_q(k-1),$$

where $P_q(k)$ denotes the number of (unordered) partitions of a positive integer k into q parts.

Example 3.3.

$$\omega_{(5,6,7)} = \omega(e^5 \wedge e^6 \wedge e^7) = e^5 \wedge e^6 \wedge e^7 - e^4 \wedge e^6 \wedge e^8 + (e^3 \wedge e^6 + e^4 \wedge e^5) \wedge e^9 -$$

$$- (e^2 \wedge e^6 + 2e^3 \wedge e^5) \wedge e^{10} + (3e^2 \wedge e^5 + 2e^3 \wedge e^4) \wedge e^{11} - 5e^2 \wedge e^4 \wedge e^{12} + 5e^2 \wedge e^3 \wedge e^{13}.$$

Proposition 3.4. *It follows from (6) that*

$$\omega(e^2 \wedge e^3 \wedge \cdots \wedge e^i \wedge e^{i+1}) \wedge \omega_I = \omega(e^2 \wedge e^3 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge e^I).$$

4. The spectral sequence and $H^*(\mathfrak{m}_0, \mathfrak{m}_0)$

In this section we compute $H^*(\mathfrak{m}_0, \mathfrak{m}_0)$.

Theorem 4.1. *The bigraded cohomology $H^*(\mathfrak{m}_0, \mathfrak{m}_0) = \oplus_{p,k} H^p_k(\mathfrak{m}_0, \mathfrak{m}_0)$ is an infinite dimensional linear space of formal series $\sum_{J,s} \alpha_{J,s}\Psi_{J,s}$ of the following infinite system $\{\Psi_{J,s}\}$ of homogeneous cocycles:*

$$\Psi_{1,1} = e_1 \otimes e^1 + \sum_{j=3}^{\infty}(j-2)e_j \otimes e^j,\ \Psi_{1,2} = e_2 \otimes e^1,\ \Psi_{2,l+2} = \sum_{j=2}^{\infty} e_{l+j} \otimes e^j,\ l \geq 0, l \neq 1,$$

$$\Psi_{I,r} = \sum_{j=0}^{\infty} e_{r+j} \otimes D^j_{-1}\omega_I,\ r{\geq}2,\ I{=}(i_1, \ldots, i_q, i_q+1), \quad q{\geq}1, \tag{7}$$

$$2 \leq i_1{<}\ldots{<}i_q,\ i_{r-2} > r-1, \text{ if } \quad 3 \leq r \leq q+1, r = q+3.$$

where ω_I stands for a basic scalar cocycle defined by (5) and D_{-1} is the operator defined by (4).

The homogeneous cocycles $\Psi_{J,s}$ have the following gradings:

$$\Psi_{1,1}, \Psi_{2,2} \in H^1_0(\mathfrak{m}_0, \mathfrak{m}_0),\ \Psi_{2,1} \in H^1_1(\mathfrak{m}_0, \mathfrak{m}_0),\ \Psi_{2,l+2} \in H^1_l(\mathfrak{m}_0, \mathfrak{m}_0),$$

$$\Psi_{I,r} \in H^{q+1}_k(\mathfrak{m}_0, \mathfrak{m}_0),\ I{=}(i_1, \ldots, i_q, i_q+1),\ k = r - (i_1{+}i_2{+}\ldots{+}i_{q-1}{+}2i_q{+}1)\,. \tag{8}$$

The cocycle $\Psi_{I,r}$ with $I{=}(i_1, i_2, \ldots, i_q, i_q+1)$ is uniquely determined by the following condition:

$$\Psi_{I,r}\left(e_{i_1}, e_{i_2}, \ldots, e_{i_q}, e_{i_q+1}\right) = e_r, \quad 2 \leq i_1{<}\ldots{<}i_q,$$

$$\Psi_{I,r}\left(e_{j_1}, e_{j_2}, \ldots, e_{j_q}, e_{j_q+1}\right) = 0, \quad 2 \leq j_1{<}\ldots{<}j_q,\ (j_1, \ldots, j_q, j_q+1) \neq I. \tag{9}$$

Proof. We consider the spectral sequence $E_r^{p,q}$ from the Proposition 2.2. Namely it follows that

$$E_1^{p,q} = (\mathfrak{m}_0)_q \otimes H^{p+q}(\mathfrak{m}_0).$$

Proposition 4.2. *1) The first differential $d_1 : E_1^{p,q} \to E_1^{p,q+1}$ is non trivial only in the cases:*

$$d_1(e_q) = -e_{q+1} \otimes e^1,\ q \geq 2.$$

2) For the second differential $d_2 : E_2^{p,q} \to E_2^{p-1,q+2}$ we have the following property:

$$d_2(e_1) = e_3 \otimes e^2, \quad d_2(e_1 \otimes \omega) = e_3 \otimes e^2 \wedge \omega, \quad d_2(e_q \otimes \omega) = 0,\ q \geq 2.$$

Proof. First of all by the definiton of $d : C^0(\mathfrak{g},\mathfrak{g})=\mathfrak{g} \to C^1(\mathfrak{g},\mathfrak{g})=\mathfrak{g}\otimes\mathfrak{g}^*$ we have

$$dX(Y) = [X,Y],\ X,Y \in \mathfrak{g}.$$

Hence for an arbitrary closed ω we have

$$\begin{gathered} de_1 = \sum_{j=2}^{\infty} e_{j+1} \otimes e^j, \quad de_j = -e_{j+1} \otimes e^1,\ j > 1; \\ d(e_1 \otimes \omega) = e_3 \otimes e^2 \wedge \omega + \dots, \quad d(e_q \otimes \omega) = e_{q+1} \otimes e^1 \wedge \omega, q > 1. \end{gathered} \tag{10}$$

We recall that the product $e^1 \wedge \omega$ is always trivial in cohomology $H^*(\mathfrak{m}_0,\mathfrak{m}_0)$ and we can shift our form $e_q \otimes \omega$ by $e_{q+1} \otimes D_{-1}\omega$ and see that

$$d(e_q \otimes \omega + e_{q+1} \otimes D_{-1}\omega) = -e_{q+2} \otimes e^1 \wedge D_{-1}\omega + \dots$$

We put everywhere dots instead of terms of higher filtration. $\square$

It follows from the Proposition 3.4 that the class $e^2 \wedge \omega$ is trivial if and only if $\omega = e^2 \wedge \tilde{\omega}$, for some $\tilde{\omega}$. We came to the following corollary:

Corollary 4.3. *The following classes in E_1 survive to E_3:*

$$\begin{gathered} e_1 \otimes e^1, \quad e_1 \otimes e^2 \wedge \omega, \quad e_2 \otimes e^1, \quad e_q \otimes e^2, \; q \geq 2 \\ e_r \otimes \omega_I, \; r \geq 2, (r, I) \neq (3, (2, i_2, \dots, i_q, i_q + 1)). \end{gathered} \tag{11}$$

Proposition 4.4. *The differential $d_s, s \geq 3$, can be non trivial only in the following case:*

$$d_s(e_1 \otimes e^2 \wedge e^3 \wedge \dots \wedge e^{s-1} \wedge \omega) = (-1)^s e_{s+1} \otimes e^2 \wedge e^3 \wedge \dots \wedge e^{s-1} \wedge e^s \wedge \omega.$$

Proof. A cocycle $\Psi_{I,r}$ (17) represents $e_r \otimes \omega_I$ and obviously $d_s(\Psi_{I,r}) = 0, s \geq 2$. Class $e_1 \otimes \omega$ survives to E_∞ if and only if ω is divisible by $e^2 \wedge e^3 \wedge \dots \wedge e^s$ for arbitrary $s \geq 2$. Hence no one class of the form $e_1 \otimes \omega$ can survive to E_∞.

From the another hand we have to take the quotient of E_s over the subspace of classes of the form $e_{s+1} \otimes e^2 \wedge e^3 \wedge \dots \wedge e^s \wedge \tilde{\omega}, s \geq 2$. It means that we have to remove from our final list the cocycles $\Psi_{I,s+1}$ with I such as $i_1 = 2, \dots, i_{s-1} = s$ if $s - 1 < q$ as well as the cocycles $\Psi_{(2,3,\dots,s),s+1}$. □

Now we leave to the reader to prove the property (9) of the basic cocycles $\Psi_{I,r}$. □

Remark. It follows from the theorem 6.1 that zero-cohomology of $\mathfrak{m}_0$ is trivial:

$$H^0(\mathfrak{m}_0, \mathfrak{m}_0) = 0.$$

One-dimensional cohomology $H^1(\mathfrak{m}_0,\mathfrak{m}_0)$ is infinite dimensional, but its homogeneous components are finite dimensional ones:

$$
\begin{gathered}
H^1_k(\mathfrak{m}_0,\mathfrak{m}_0)=0, k\le -1,\ H^1_0(\mathfrak{m}_0,\mathfrak{m}_0)=\langle \Psi_{1,1},\Psi_{2,2}\rangle,\\
H^1_1(\mathfrak{m}_0,\mathfrak{m}_0)=\langle \Psi_{1,2}\rangle,\ H^1_k(\mathfrak{m}_0,\mathfrak{m}_0)=\langle \Psi_{2,k+2}\rangle,\ k\ge 2.
\end{gathered}
\tag{12}
$$

The structure of $H^1(\mathfrak{m}_0,\mathfrak{m}_0)$ was the subject in [4], but it was found earlier in [8, 9] (for instance our cocycles $\Psi_{1,1},\Psi_{2,2}$ coinside with ω_1,ω_2 in [4] and they are also equal to d_1 and d_2-2d_1 in [8, 9]). Let us recall that an arbitrary positively graded Lie algebra $\mathfrak{g}=\oplus_i\mathfrak{g}_i$ has a derivation τ defined by means of its graded structure

$$
\tau(X)=iX,\ X\in\mathfrak{g}_i.
$$

In our case $\tau=\Psi_{1,1}+2\Psi_{2,2}$. Another obvious fact is that the one-dimensional cohomology $H^1(\mathfrak{g},\mathfrak{g})$ endowed with the Nijehhuis-Richardson bracket is isomorphic to the algebra of outer derivations of $\mathfrak{g}$.

Example 4.5. We have defined the two-dimensional cocycle $\Psi_{i,i+1,r}$ by (18) as

$$
\Psi_{i,i+1,r}=\sum_{l=0}^{\infty}e_{r+l}\otimes D_{-1}^l\omega_{i,i+1}.
$$

An elementary exercise on the properties of the operator D_{-1} will be to verify that

$$
D_{-1}^l\omega_{i,i+1}=\sum_{s=0}^{i-2}(-1)^s\binom{l+s}{s}e^{i-s}\wedge e^{i+1+s+l}.
$$

Hence we have the following formula

$$\Psi_{i,i+1,r} = \sum_{l=0}^{\infty} \sum_{s=0}^{i-2} (-1)^s \binom{l+s}{s} e_{r+l} \otimes e^{i-s} \wedge e^{i+1+s+l}. \tag{13}$$

It means that

$$\Psi_{i,i+1,r}(e_k, e_m) = (-1)^{i-k} \binom{m-i-1}{i-k} e_{m+k-2i-1},\ 2 \le k \le i < m.$$

Hence the cocycles $\Psi_{i,i+1,r}$ coinside with the basic cocycles $\Psi_{i,r}$ considered by Khakimdjanov in [9] where he used Vergne's general algorithm [11].

It is evident that a homogeneous subspace $H^2_k(\mathfrak{m}_0, \mathfrak{m}_0)$ is infinite dimensional as it was remarked in [4].

Example 4.6. The infinite dimensional $H^3(\mathfrak{m}_0, \mathfrak{m}_0)$ is the space of formal series of basic cocycles

$$\Psi_{(j,i,i+1),r} = e_r \otimes \omega(e^j \wedge e^i \wedge e^{i+1}) + e_{r+1} \otimes D_{-1}\omega(e^j \wedge e^i \wedge e^{i+1}) + \ldots, r \ge 2,\ 2 \le j < i,$$

where the cocycles $\left\{\Psi_{(2,i,i+1),3}, i > 2\right\}$, and $\Psi_{(2,3,4),5}$ have been removed from the list of basic cocycles according to the rule from (17).

5. The scalar cohomology $H^*(\mathfrak{m}_2)$

The operator $D_2 = ad^* e_2 : \Lambda^*(\mathfrak{b}) \to \Lambda^*(\mathfrak{b})$ inducing $\mathcal{D}_2 : H^*(\mathfrak{b}) \to H^*(\mathfrak{b})$ can be defined by

$$\begin{gathered} D_2(e^1) = D_2(e^4) = 0,\ D_2(e^3) = e^1,\ D_2(e^i) = e^{i-2},\ i \ge 5, \\ D_2(\xi \wedge \eta) = D_2(\xi) \wedge \eta + \xi \wedge D_2(\eta),\ \forall \xi, \eta \in \Lambda^*(\mathfrak{b}). \end{gathered} \tag{14}$$

It is immediate that

$$\begin{gathered}\mathcal{D}_2(e^3) = e^1, \quad \mathcal{D}_2(e^1) = 0,\\ \mathcal{D}_2(\omega_{\mathfrak{b}}(e^3 \wedge e^4)) = 0, \mathcal{D}_2(\omega_{\mathfrak{b}}(e^k \wedge e^{k+1})) = -2\omega_{\mathfrak{b}}(e^{k-1} \wedge e^k),\ k \geq 4.\end{gathered} \tag{15}$$

Proposition 5.1. *Let $3 \leq i_1 < \dots < i_{p-1} < i$ and $\xi = e^{i_1} \wedge \ldots \wedge e^{i_{p-1}}$, then*

$$\mathcal{D}_2(\omega_{\mathfrak{b}}(\xi \wedge e^i \wedge e^{i+1})) = \omega_{\mathfrak{b}}((D_2 + D_1^2)(\xi) \wedge e^i \wedge e^{i+1}) - 2\omega_{\mathfrak{b}}(\xi \wedge e^{i-1} \wedge e^i).$$

Theorem 5.2. *The bigraded cohomology algebra $H^*(\mathfrak{m}_2) = \oplus_{q,k} H^q_k(\mathfrak{m}_2)$ is spanned by cohomology classes of the following homogeneous cocycles:*

$$e^1,\ e^2,\ e^2 \wedge e^3,\ e^3 \wedge e^4 - e^2 \wedge e^5, \tag{16}$$

$$w_{i_1,\dots,i_q,i_q+1,i_q+2} = \sum_{l \geq 1} \frac{1}{2^l} \omega \left((D_2 + D_1^2)^l (e^{i_1} \wedge \ldots \wedge e^{i_q}) \wedge e^{i_q+1+l} \wedge e^{i_q+2+l} \right),$$

where $1 \leq q,\ 3 \leq i_1 < i_2 < \dots < i_q$, in particular for $q \geq 3$,

$$\dim H^q_{k+\frac{q(q+1)}{2}}(\mathfrak{m}_2) = P_q(k) - P_q(k-1) - P_q(k-2) + P_q(k-3).$$

Example 5.3.

$$w_{5,6,7} = \omega(e^5 \wedge e^6 \wedge e^7) + \omega(e^3 \wedge e^7 \wedge e^8) = e^5 \wedge e^6 \wedge e^7 + (e^3 \wedge e^7 - e^4 \wedge e^6) \wedge e^8 +$$

$$+ (e^4 \wedge e^5 - e^2 \wedge e^7) \wedge e^9 + (e^2 \wedge e^6 - e^3 \wedge e^5) \wedge e^{10} + e^3 \wedge e^4 \wedge e^{11} - e^2 \wedge e^4 \wedge e^{12} + e^2 \wedge e^3 \wedge e^{13}.$$

1) The space $H^2(\mathfrak{m}_2)$ is two-dimensional and it is spanned by the cohomology classes represented by cocycles $e^2 \wedge e^3$ and $e^3 \wedge e^4 - e^2 \wedge e^5$ of second gradings 5 and 7 respectively;

2) $H^3(\mathfrak{m}_2)$ is infinite dimensional and it is spanned by

$$w_{k,k+1,k+2} = \sum_{l \geq 0} \omega\left(e^{k-2l} \wedge e^{k+1+l} \wedge e^{k+2+l}\right),\ k \geq 3.$$

6. Adjoint cohomology $H^*(\mathfrak{m}_2, \mathfrak{m}_2)$

Theorem 6.1. *The bigraded cohomology $H^*(\mathfrak{m}_2, \mathfrak{m}_2) = \oplus_{p,k} H^p_k(\mathfrak{m}_2, \mathfrak{m}_2)$ is an infinite dimensional linear space of formal series $\sum_{J,s} \alpha_{J,s} \Phi_{J,s}$ of the following infinite system $\{\Phi_{J,s}\}$ of homogeneous cocycles:*

(17)

$$\Phi_{1,1} = \sum_{j=1}^{\infty} j e_j \otimes e^j,\ \Phi_{2,l+2} = \sum_{j=2}^{\infty} e_{l+j} \otimes e^j,\ l \geq 3, \quad \Phi_{2,3,m} = \sum_{j=0}^{\infty} e_{m+j} \otimes e^2 \wedge e^{3+j},\ m = 3, m \geq 7,$$

$$\Phi_{2,3,1} = e_1 \otimes e^2 \wedge e^3 + \frac{1}{2} \sum_{j=0}^{\infty} e_{5+j} \otimes \left(e^4 \wedge e^{5+j} - (j+1) e^3 \wedge e^{6+j} + \frac{(j+2)(j+1)}{2} e^2 \wedge e^{7+j}\right),$$

$$\Phi_{2,3,2} = \sum_{i=0}^{\infty} e_{2+i} \otimes e^2 \wedge e^{3+i} + \frac{1}{2} \sum_{j=0}^{\infty} e_{6+j} \otimes \left(e^4 \wedge e^{5+j} - (j+1) e^3 \wedge e^{6+j} + \frac{(j+2)(j+1)}{2} e^2 \wedge e^{7+j}\right),$$

$$\Phi_{3,4,l} = \sum_{i=0}^{\infty} e_{l+i} \otimes \left(e^3 \wedge e^{4+i} - (i+1) e^2 \wedge e^{5+i}\right),\ l \geq 3,$$

$$\Phi_{I,r} = \sum_{j=0}^{\infty} e_{r+j} \otimes \tilde{D}^j_{-1} w_I,\ r \geq 3,\ I = (i_1, \ldots, i_q, i_q+1, i_q+2), \quad q \geq 1, \quad 3 \leq i_1 < \ldots < i_q,$$

$$\text{if } \ r = 4, q \geq 2, \ \text{ then } \ i_1 > 3,$$

$$\text{if } \ 5 \leq r \leq q+3,\ r = q+6, \ \text{ then } i_{r-4} > r-1, \ \text{ or } \ i_{r-4} = r-1,\ i_1 > 3,$$

$$\text{if } \ r = q+5, \ \text{ then } \ i_{r-3} > r-1.$$

with w_I *stands for a basic scalar cocycle defined by (16),* $\tilde{D}^j_{-1}w_I$ *defined by induction by* $d\tilde{D}^j_{-1}w_I = e^1 \wedge \tilde{D}^{j-1}_{-1}w_I + e^2 \wedge D^{j-2}_{-1}w_I$ *and* $d\tilde{D}_{-1}w_I = e^1 \wedge \tilde{w}_I$.

The homogeneous cocycles $\Phi_{J,s}$ *have the following gradings:*

$$\begin{gathered}\Phi_{1,1} \in H^1_0(\mathfrak{m}_2, \mathfrak{m}_2),\ \Phi_{2,l+2} \in H^1_l(\mathfrak{m}_2, \mathfrak{m}_2), \Phi_{2,3,l} \in H^2_{l-5}(\mathfrak{m}_2, \mathfrak{m}_2), \Phi_{3,4,l} \in H^2_{l-7}(\mathfrak{m}_2, \mathfrak{m}_2),\\ \Phi_{I,r} \in H^{q+2}_k(\mathfrak{m}_0, \mathfrak{m}_0),\ I{=}(i_1, \dots, i_q, i_q+1, i_q+2),\ k = r - (i_1{+}i_2{+}\dots{+}i_{q-1}{+}3i_q{+}3)\,.\end{gathered} \tag{18}$$

The cocycle $\Phi_{I,r}$ *with* $I{=}(i_1, i_2, \dots, i_q, i_q+1, i_q+2)$ *is uniquely determined by the following condition:*

$$\begin{gathered}\Phi_{I,r}\left(e_{i_1}, e_{i_2}, \dots, e_{i_q}, e_{i_q+1}, e_{i_q+2}\right) = e_r, \quad 2 \le i_1 {<} \dots {<} i_q,\\ \Phi_{I,r}\left(e_{j_1}, e_{j_2}, \dots, e_{j_q}, e_{j_q+1}, e_{j_q+2}\right) = 0, \quad 2 \le j_1 {<} \dots {<} j_q,\ (j_1, \dots, j_q, j_q+1, j_q+2) \neq I.\end{gathered} \tag{19}$$

Proof. We consider the spectral sequence considered above. It follows from the Proposition 2.2 that

$$E_1^{p,q} = (\mathfrak{m}_2)_q \otimes H^{p+q}(\mathfrak{m}_2).$$

Proposition 6.2. *1) The first differential* $d_1 : E_1^{p,q} \to E_1^{p,q+1}$ *is non trivial only in the cases:*

$$d_1(e_q) = -e_{q+1} \otimes e^1,\ q \ge 2.$$

2) The second differential $d_2 : E_2^{p,q} \to E_2^{p-1,q+2}$ *has the only one non trivial value:*

$$d_2(e_1) = e_3 \otimes e^2.$$

Proof. We start with

(20)
$$de_1 = \sum_{j=2}^{\infty} e_{j+1} \otimes e^j, \quad de_2 = -e_3 \otimes e^1 + \sum_{j=2}^{\infty} e_{j+2} \otimes e^j, \quad de_j = -e_{j+1} \otimes e^1 - e_{j+2} \otimes e^2,\ j \geq 3;$$

$$d(e_1 \otimes \omega) = e_3 \otimes e^2 \wedge \omega + \dots, \quad d(e_q \otimes \omega) = e_{q+1} \otimes e^1 \wedge \omega + \dots$$

We recall that the products $e^1 \wedge \omega$ and $e^2 \wedge \omega$ and are always trivial in cohomology $H^*(\mathfrak{m}_2)$ for an arbitrary closed form ω. Also one can remark that a form $e^1 \wedge \xi_1 + e^2 \wedge \xi_2$ is closed if and anly if it is exact. Hence we can shift our form $e_q \otimes \omega$ by $e_{q+1} \otimes \tilde{D}_{-1}\omega$, where $d\tilde{D}_{-1}\omega = e^1 \wedge \omega$ and see that

$$d(e_q \otimes \omega + e_{q+1} \otimes \tilde{D}_{-1}\omega) = -e_{q+2} \otimes \left(e^1 \wedge \tilde{D}_{-1}\omega + e^2 \wedge \omega\right) + \dots$$

We put everywhere dots instead of terms of higher filtration. □

It follows from the Proposition 3.4 that the class $e^2 \wedge \omega$ is trivial if and only if $\omega = e^2 \wedge \tilde{\omega}$, for some $\tilde{\omega}$. We came to the following corollary:

Corollary 6.3. *The following classes in E_1 survive to E_3:*

(21)
$$e_1 \otimes e^1, \quad e_2 \otimes e^1, \quad e_1 \otimes e^2, \quad e_2 \otimes e^2, \quad e_q \otimes e^2,\ q \geq 4,$$
$$e_l \otimes e^2 \wedge e^3, e_l \otimes \left(e^3 \wedge e^4 - e^2 \wedge e^5\right), l \geq 1, \quad e_r \otimes w_I,\ r \geq 1.$$

Proposition 6.4. *The differentials $d_s, 3 \leq s \leq 4$ are non trivial in the following cases:*

$$d_3(e_1 \otimes e^2) = -e_4 \otimes e^2 \wedge e^3,\ d_3(e_1 \otimes w_{i_1,i_2,i_3,\dots,i_q+2}) = e_4 \otimes w_{3,i_1,i_2,i_3,\dots,i_q+2}, \tag{22}$$

$$d_3(e_2 \otimes e^2) = -e_5 \otimes e^2 \wedge e^3,\ d_3(e_2 \otimes w_{i_1,i_2,i_3,\dots,i_q+2}) = e_5 \otimes w_{3,i_1,i_2,i_3,\dots,i_q+2}, i_1 > 3,$$

$$d_4(e_2 \otimes e^1) = 2e_6 \otimes e^2 \wedge e^3,\ d_4(e_2 \otimes w_{3,i_2,i_3,\dots,i_q+2}) = -e_6 \otimes w_{3,4,i_2,i_3,\dots,i_q+2}, i_2 > 4.$$

Proof. We will prove some of the formulas (22), the rest of them can be obtained analogously

(23)

$$d(e_1 \otimes e^2) = e_4 \otimes e^3 \wedge e^2 + \dots, \quad d(e_2 \otimes e^2 + e_3 \otimes e^3 + e_4 \otimes e^4) = e_5 \otimes \left(-2e^2 \wedge e^3 - e^1 \wedge e^4\right) + \dots,$$

$$d(e_2 \otimes e^1 + e_5 \otimes e^4) = -2e_6 \otimes e^1 \wedge e^4 + \dots,$$

$$d(e_1 \otimes w_{i_1,\dots,i_q+2} + e_3 \otimes \xi) = e_4 \otimes (e^3 \wedge w_{i_1,\dots,i_q+2} - e^1 \wedge \xi) + \dots =$$

$$= e_4 \otimes (e^3 \wedge w_{i_1,\dots,i_q+2} + e^2 \wedge D_2 D_{-1}\xi - dD_{-1}\xi) + \dots, \quad d\xi = e^2 \wedge w_{i_1,\dots,i_q+2}.$$

It is easy to see that in the decomposition $e^3 \wedge w_{i_1,\dots,i_q+2} + e^2 \wedge D_2 D_{-1}\xi$ we have the only one monomial $e^3 \wedge e^{i_1} \wedge \dots \wedge e^{i_q} \wedge e^{i_q+1} \wedge e^{i_q+2}$ with neighboring three last superscripts and it follows that the expression is cohomologous to $w_{3,i_1,\dots,i_q+2}$ if $i_1 > 3$ and it is cohomologous to zero if $i_1 = 3$. $\square$

Proposition 6.5. *The differential d_5 is non trivial in the following cases:*

$$d_5\left(e_1\otimes(e^3\wedge e^4-e^2\wedge e^5)\right)=e_6\otimes w_{3,4,5},\quad d_5\left(e_2\otimes(e^3\wedge e^4-e^2\wedge e^5)\right)=e_7\otimes w_{3,4,5}, \tag{24}$$

$$d_5\left(e_1\otimes w_{3,i_2,i_3,\dots,i_q+2}\right)=-e_6\otimes w_{3,5,i_2,i_3,\dots,i_q+2},\, i_2>5,$$

$$d_5\left(e_2\otimes w_{3,4,i_3,\dots,i_q+2}\right)=e_7\otimes w_{3,4,5,i_3,\dots,i_q+2},\, i_3>5.$$

Proof. Again we will not prove all the formulas (24), leaving the rest of them to the reader. For instance

$$d(e_1\otimes(e^3\wedge e^4-e^2\wedge e^5)+\frac{1}{2}e_3\otimes(e^4\wedge e^5-e^3\wedge e^6+e^2\wedge e^7)+\frac{1}{2}e_4\otimes(e^4\wedge e^6-2e^3\wedge e^7+3e^2\wedge e^8)+ \tag{25}$$

$$+\frac{1}{2}e_5\otimes(e^5\wedge e^6-2e^3\wedge e^8+5e^2\wedge e^9))=e_6\otimes(e^5\wedge e^3\wedge e^4-\frac{1}{2}e^2\wedge(e^4\wedge e^6-2e^3\wedge e^7)-\frac{1}{2}e^2\wedge e^4\wedge e^6-d\xi)+\dots,$$

$$\xi=\frac{1}{2}(e^5\wedge e^7-e^4\wedge e^8-e^3\wedge e^9+6e^2\wedge e^{10}),\; w_{3,4,5}=e^3\wedge e^4\wedge e^5-e^2\wedge e^4\wedge e^6+e^2\wedge e^3\wedge e^7.$$

$\square$

Proposition 6.6. *The differential $d_s, s \geq 6$ can be non trivial only at the classes of the form $e_1 \otimes w_I$ or $e_2 \otimes w_J$ and only in the following cases:*

$$J = (j_1, \dots, j_q, j_q+1, j_q+2), j_1 = 3, j_2 = 4, \dots, j_{s-3} = s-1, \;\; q > s-3, \; j_q > s; \tag{26}$$

$$I = (i_1, \dots, i_q, i_q+1, i_q+2), \; i_1 = 3, i_{s-4} = s-1, \;\; q > s-4, \; i_q > s;$$

$$I = (i_1, \dots, i_q, i_q+1, i_q+2), \; i_1 = 3, i_{s-4} = s-2, \;\; q > s-3, \; i_q > s;$$

$$I = J = (3, 4, 5, \dots, s-1).$$

Proof. We will sketch the proof of this proposition in the spirit of previous two propositions. The idea is the same: one has to keep an eye only on the monomials of the form $e^{i_1}\wedge \dots \wedge e^{i_q} \wedge e^{i_q+1} \wedge e^{i_q+2}$ in the decomposition of scalar cocycles. □

Now a few words about forms $\tilde{D}^j_{-1}w$. We give no explicit expressions for them like in the case of $\mathfrak{m}_0$. However it is possible to write them out. For instance one can define $\tilde{D}_{-1}w_{i_1,\dots,i_q+2}$ in a following way:

$$\tilde{D}_{-1}w_{i_1,\dots,i_q+2} = D_{-1}w_{i_1,\dots,i_q+2} - \sum_{s=0} \frac{(s+1)}{2^s}\omega\left((D_1^2+D_2)^s D_1(e^{i_1}\wedge \dots \wedge e^{i_q})\wedge e^{i_q+2+s}\wedge e^{i_q+3+s}\right).$$

Now the next step is to define a form $\tilde{D}^2_{-1}w$ such that

$$d\tilde{D}^2_{-1}w = e^1 \wedge \tilde{D}_{-1}w + e^2 \wedge w. \tag{27}$$

On the right hand side we have a closed form and it has the form $e^1 \wedge \xi + e^2 \wedge \eta$. Hence it is the exact form and we will denote by $\tilde{D}^2_{-1}w$ an arbitrary form satisfying (27).

Now we come to the inductive procedure. Let us assume that we have found forms $\tilde{D}^{j-1}_{-1}w$ and $\tilde{D}^{j-2}_{-1}w$ such that

$$d\tilde{D}^{j-1}_{-1}w = e^1 \wedge \tilde{D}^{j-2}_{-1}w + e^2 \wedge D^{j-3}_{-1}w,\ d\tilde{D}^{j-2}_{-1}w = e^1 \wedge \tilde{D}^{j-3}_{-1}w + e^2 \wedge D^{j-4}_{-1}w.$$

Then it is easy to see that the form $e^1 \wedge \tilde{D}^{j-1}_{-1}w + e^2 \wedge \tilde{D}^{j-2}_{-1}w$ is closed and hence it is exact. We can define $D^j_{-1}w$ such that

$$d\tilde{D}^j_{-1}w = e^1 \wedge \tilde{D}^{j-1}_{-1}w + e^2 \wedge D^{j-2}_{-1}w.$$

Now it is evident that the element

$$\Phi_{J,r} = \sum_{j=0}^{+\infty} e_{r+j} \otimes \tilde{D}^j_{-1}w_J,\ J = (j_1, \dots, j_q, j_q+1, j_q+2),$$

is closed element in $C^{q+2}(\mathfrak{m}_0, \mathfrak{m}_0)$ and represents the class $e_r \otimes w_J$ in E_1 term. $\square$

Example 6.7. The infinite basis of $H^2(\mathfrak{m}_2, \mathfrak{m}_2)$ consists of the following cocycles:

$$\Phi_{2,3,m} = \Psi_{2,3,m},\ m = 3, m \geq 7,\ \Phi_{3,4,l} = \Psi_{3,4,l},\ l \geq 3, \tag{28}$$

$$\Phi_{2,3,1} = e_1 \otimes e^2 \wedge e^3 + \frac{1}{2}\Psi_{4,5,5},\ \Phi_{2,3,2} = \Psi_{2,3,2} + \frac{1}{2}\Psi_{4,5,6}.$$

It was established by Vergne that an arbitrary two-dimensional adjoint cocycle of $\mathfrak{m}_2$ vanishing at e_1 is determined by its values on pairs e_2, e_3 and e_3, e_4 respectively [11]. But we see that some of them, namely $\Psi_{2,3,m},\ m = 4, 5, 6$, are coboundaries.

Example 6.8. The infinite dimensional $H^3(\mathfrak{m}_0, \mathfrak{m}_0)$ is the space of formal series of basic cocycles

$$\Phi_{(i,i+1,i+2),r} = e_r \otimes w_{i,i+1,i+2} + e_{r+1} \otimes \tilde{D}_{-1} w_{i,i+1,i+2} + \ldots, r \geq 3,\ 3 \leq i,$$

where we have removed the cocycles $\Phi_{(3,4,5),6}$ and $\Phi_{(3,4,5),7}$ from the list.

Corollary 6.9. *1)* $H^0(\mathfrak{m}_2, \mathfrak{m}_2) = 0$.

2) $H^1(\mathfrak{m}_2, \mathfrak{m}_2)$ *is infinite dimensional and*

$$\dim H^1_q(\mathfrak{m}_2, \mathfrak{m}_2) = \begin{cases} 0, \ q \leq -1 \quad \text{or} \quad q = 1, \\ 1, \quad q \geq 2 \quad \text{or} \quad q = 0. \end{cases}$$

3) $H^2(\mathfrak{m}_2, \mathfrak{m}_2)$ *is infinite dimensional and*

$$\dim H^2_q(\mathfrak{m}_2, \mathfrak{m}_2) = \begin{cases} 0, \ q \leq -5, \\ 1, \ q = -1, 0, 1, \\ 2, \ q \geq 2 \quad \text{or} \quad q = -4, -3, -2. \end{cases}$$

Remark. One-dimensional cohomology $H^1(\mathfrak{m}_2, \mathfrak{m}_2)$ was also found in [11, 8, 9] and rediscovered later in [5]. The property of $\mathfrak{m}_2$ that $H^2_q(\mathfrak{m}_2, \mathfrak{m}_2) = 0, q \leq -5$ was established in [4]. We corrected in the present article the values of dimensions $\dim H^2_q(\mathfrak{m}_2, \mathfrak{m}_2)$ from [5] for $q = -1, 0, 1$: they are all equal to one.

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